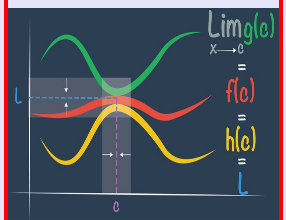
Math 261

Spring 2022

Lecture 8



Class QZ 4

For $\varepsilon > 0$, find $0 < \delta \le 1$ such that

$$\lim_{x \to -1} (x^2 - 3x) = 4 \checkmark$$

$f(x) = x^2 - 3x$

$L = 4 \checkmark$

$a = -1$

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$|x^2 - 3x - 4| < \varepsilon$ $\quad$ $|x + 1| < \delta$

$|(x-4)(x+1)| < \varepsilon$

$\underbrace{|x-4|}_{\text{Bound}} \; \underbrace{|x+1|}_{\text{keep}} < \varepsilon$

$6\,|x+1| < \varepsilon$

$|x+1| < \frac{\varepsilon}{6}$

$|x+1| < 1$

$-1 < x + 1 < 1$

$-2 < x < 0$

$-6 < x - 4 < -4 < 6$

$-6 < x - 4 < 6$

$|x - 4| < 6$

$\boxed{\text{Pick } \delta = \min\left\{1, \frac{\varepsilon}{6}\right\}}$

For the function $f(x)$

First derivative is $f'(x)$ "f-prime of x"

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ if this limit exists.

Find $f'(x)$ using definition of $f(x) = 2x^2 - 3x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{2(x+h)^2 - 3(x+h) - (2x^2 - 3x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 \cancel{-3x} - 3h \cancel{-2x^2} \cancel{+3x}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(4x + 2h - 3)}{\cancel{h}} = \lim_{h \to 0} (4x + 2h - 3) = \boxed{4x - 3}$$

$f(x) = 2x^2 - 3x$ $\quad$ $f'(x) = 4x - 3$

Find $f'(\frac{1}{2})$ for $f(x) = \frac{1}{x^2}$ using definition of $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\boxed{\frac{1}{(x+h)^2}} - \boxed{\frac{1}{x^2}}}{h}$$

LCD $= x^2(x+h)^2$

$$= \lim_{h \to 0} \frac{x^2 - (x+h)^2}{h x^2 (x+h)^2} = \lim_{h \to 0} \frac{\cancel{x^2} \cancel{- x^2} - 2xh - h^2}{h x^2 (x+h)^2}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(-2x - h)}{\cancel{h} x^2 (x+h)^2} = \lim_{h \to 0} \frac{-2x - h}{x^2 (x+h)^2} = \frac{-2x - 0}{x^2 (x+0)^2}$$

$$= \frac{-2x}{x^4} = \boxed{\frac{-2}{x^3}}$$

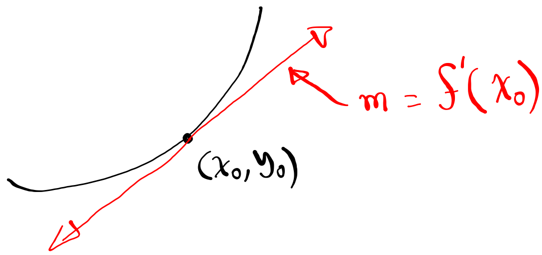
$f(x) = \frac{1}{x^2}$ $\quad$ $f'(x) = \frac{-2}{x^3}$ $\quad$ $f'(\frac{1}{2}) = \frac{-2}{(\frac{1}{2})^3} = \frac{-2}{\frac{1}{8}}$

$\delta = \frac{.1}{7}$

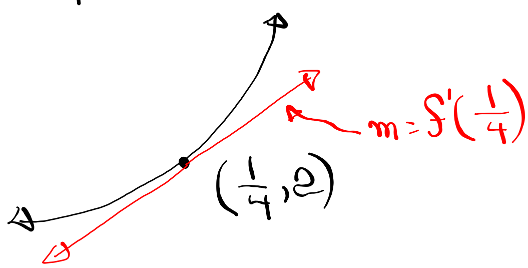
$\boxed{f'(\frac{1}{2}) = -16}$

March 7, 2022

Slope of the tangent line at (x_0, y_0)



Find slope of the tangent line at $x = \frac{1}{4}$ for the function $f(x) = \frac{1}{\sqrt{x}}$.



$$f\left(\tfrac{1}{4}\right) = \frac{1}{\sqrt{\frac{1}{4}}} = \frac{1}{\frac{1}{2}} = 2$$

Find $f'(x)$ using definition for $f(x) = \frac{1}{\sqrt{x}}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} = \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} \cdot \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}}$$

$$= \lim_{h\to 0} \frac{\overset{\cancel{x} - \cancel{x} - \cancel{h}}{x - (x+h)}}{\cancel{h}\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})} = \lim_{h\to 0} \frac{-1}{\sqrt{x}\sqrt{x+\cancelto{0}{h}}(\sqrt{x} + \sqrt{x+\cancelto{0}{h}})}$$

$$= \frac{-1}{\sqrt{x}\sqrt{x}(\sqrt{x} + \sqrt{x})} = \frac{-1}{x \cdot 2\sqrt{x}} = \frac{-1}{2x\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{-\sqrt{x}}{2x^2}$$

$f(x) = \frac{1}{\sqrt{x}}$ $x > 0$ $f(x) = \frac{1}{\sqrt{x}}$ $f'(x) = \frac{-\sqrt{x}}{2x^2}$

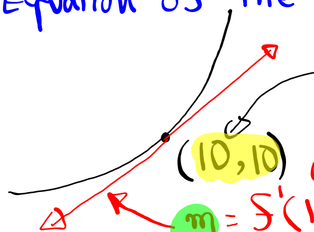
$$f'\left(\tfrac{1}{4}\right) = \frac{-\sqrt{\frac{1}{4}}}{2\left(\frac{1}{4}\right)^2} = \frac{-\frac{1}{2}}{2 \cdot \frac{1}{16}} = \frac{-\frac{1}{2}}{\frac{1}{8}} = \frac{-1}{2} \div \frac{1}{8} = \frac{-1}{2} \cdot \frac{8}{1} = \boxed{-4}$$

Given $f(x)=\frac{x}{x-9}$, $f'(x)=\frac{-9}{(x-9)^2}$

Find

1) Domain of $f(x)$ All reals except 9 $(-\infty,9)\cup(9,\infty)$

2) Equation of the tan. line at $x=10$.



$f(10)=\frac{10}{10-9}$

$=10$

$m=f'(10)=\frac{-9}{(10-9)^2}=\frac{-9}{1^2}=-9$

$y-y_1=m(x-x_1)$

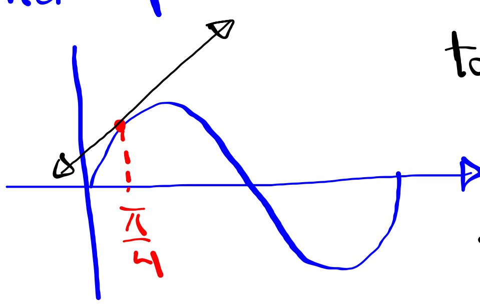
$y-10=-9(x-10)$ $\boxed{y=-9x+100}$

$f(x)=\sin x$, $f'(x)=\cos x$

$f(\frac{\pi}{4})=\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}$

$f'(\frac{\pi}{4})=\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}$

Find equation of the tan. line at $x=\frac{\pi}{4}$.



tan. Point

$\left(\frac{\pi}{4}, f(\frac{\pi}{4})\right)=\left(\frac{\pi}{4},\frac{\sqrt{2}}{2}\right)$

Slope of tan. line at $x=\frac{\pi}{4}$

$m=f'(\frac{\pi}{4})=\frac{\sqrt{2}}{2}$

$y-y_1=m(x-x_1)$

$y-\frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{2}(x-\frac{\pi}{4})$

$\boxed{y=\frac{\sqrt{2}}{2}x-\frac{\pi\sqrt{2}}{8}+\frac{\sqrt{2}}{2}}$

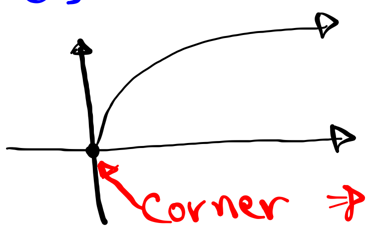
$f'(x)$ does not exist at

1) Corners

2) Vertical Tangents

3) points of discontinuity.

$f(x) = \sqrt{x}$



corner $\Rightarrow f'(x)$ does not exist at $x=0$

$f'(x) = \frac{1}{2\sqrt{x}}$ Not defined at $x=0$.

For $y = f(x)$, first derivative

y' , $f'(x)$, $\frac{d}{dx}[y]$, $\frac{d}{dx}[f(x)]$

If $f(x) = c$, $f'(x) = 0$

$\frac{d}{dx}[c] = 0$ ✓

If $f(x) = x^n$, $f'(x) = n x^{n-1}$ Power rule

$\frac{d}{dx}[x^n] = n x^{n-1}$

$\frac{d}{dx}[c f(x)] = c \frac{d}{dx}[f(x)]$

$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$

$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$

$f(x) = 3x^2 + 8$ find $f'(x)$.

$$f'(x) = \frac{d}{dx}\left[f(x)\right] = \frac{d}{dx}\left[3x^2+8\right]$$

$$= \frac{d}{dx}\left[3x^2\right] + \frac{d}{dx}\left[8\right] \to 0$$

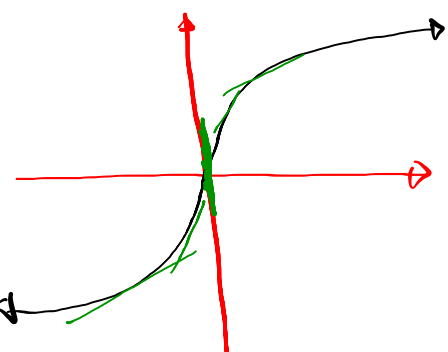
$$= 3\frac{d}{dx}\left[x^2\right] + 0$$

$$= 3 \cdot 2x^{2-1} = \boxed{6x}$$

Given $f(x) = \sqrt[3]{x}$

1) Domain $(-\infty, \infty)$

2) Rough graph



$f(x) = x^{1/3}$

3) find $f'(x)$ $f'(x) = \frac{d}{dx}\left[x^{1/3}\right] = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-\frac{2}{3}}$

4) At what point $f'(x)$ is not defined? at $x=0$

$$f'(x) = \frac{1}{3x^{2/3}} = \frac{1}{3\sqrt[3]{x^2}} \cdot \frac{\sqrt[3]{x}}{\sqrt[3]{x}} = \frac{\sqrt[3]{x}}{3\sqrt[3]{x^3}} = \boxed{\frac{\sqrt[3]{x}}{3x}}$$

Find the equation of the line line at $x=1$

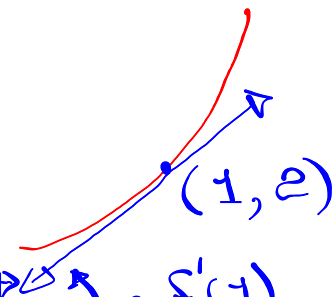
for $f(x) = x^3 + 5x^2 - 4$

$f(1) = 1^3 + 5(1)^2 - 4 = 2$

$f(x) = x^3 + 5x^2 - 4$

$f'(x) = 3x^2 + 5 \cdot 2x - 0$

$f'(x) = 3x^2 + 10x$



$m = f'(1)$

$= 3(1)^2 + 10(1)$

$= 13$

$y - y_1 = m(x - x_1)$

$y - 2 = 13(x - 1)$

$\boxed{y = 13x - 11}$

Product Rule

$$\frac{d}{dx}\left[f(x) \cdot g(x)\right] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

ex: $$\frac{d}{dx}\left[(x^2 - 5x)(x^3 + 4)\right] = (2x - 5) \cdot (x^3 + 4) + (x^2 - 5x) \cdot (3x^2 + 0)$$

$$= 2x^4 + 8x - 5x^3 - 20 + 3x^4 - 15x^2$$

$$= 5x^4 - 5x^3 - 15x^2 + 8x - 20$$

$$\frac{d}{dx}\left[\underset{f(x)}{x^6}\ \underset{g(x)}{\sin x}\right] = \underset{f'(x)}{6x^5}\cdot\underset{g(x)}{\sin x} + \underset{f(x)}{x^6}\cdot\underset{g'(x)}{\cos x}$$

$y = (4x^2-1)(4x^2+1)$

$$\frac{dy}{dx} = \frac{d}{dx}\left[(4x^2-1)(4x^2+1)\right] = 8x(4x^2+1)+(4x^2-1)\cdot 8x$$

$$= 32x^3 \cancel{+8x} + 32x^3 \cancel{-8x}$$

$$= \boxed{64x^3}$$

$y = (4x^2-1)(4x^2+1)$

$y = 16x^4 - 1$

$$\frac{dy}{dx} = \frac{d}{dx}\left[16x^4-1\right] = 16\cdot 4x^3 - 0 = \boxed{64x^3}$$

Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{[g(x)]^2}$$

Find $\frac{d}{dx}\left[\frac{2x}{x^2+1}\right] = \frac{2\cdot(x^2+1) - 2x\cdot 2x}{[x^2+1]^2}$

$$= \frac{2x^2+2-4x^2}{(x^2+1)^2} = \boxed{\frac{2-2x^2}{(x^2+1)^2}}$$

Find $\frac{d}{dx}\left[\tan x\right] = \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right]$

$$= \frac{\cancel{\frac{d}{dx}}[\sin x]\cdot\cos x - \sin x\cdot\cancel{\frac{d}{dx}}[\cos x]}{\cos^2 x}$$

$\cos x$ $-\sin x$

$$= \frac{\cos x\cdot\cos x - \sin x\cdot(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$\boxed{\frac{d}{dx}\left[\tan x\right] = \sec^2 x}$$

$$= \frac{1}{\cos^2 x} = \left[\frac{1}{\cos x}\right]^2 = \sec^2 x$$

Proving the quotient rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)\cdot g(x) - f(x)\cdot g(x+h)}{h\,g(x)\,g(x+h)}$$

$$= \lim_{h\to 0}\frac{\boxed{f(x+h)\cdot g(x) - f(x)g(x)}\ \boxed{-f(x)\cdot g(x+h) + f(x)g(x)}}{h\,g(x)\,g(x+h)}$$

$$= \lim_{h\to 0}\frac{g(x)\cdot[f(x+h) - f(x)] - f(x)[g(x+h) - g(x)]}{h\,g(x)\cdot g(x+h)}$$

Divide by h

$$= \lim_{h\to 0}\frac{g(x)\cdot\frac{f(x+h) - f(x)}{h} - f(x)\cdot\frac{g(x+h) - g(x)}{h}}{g(x)\cdot g(x+h)}$$

$$= \lim_{h \to 0} \frac{g(x) \cdot \frac{f(x+h) - f(x)}{h} - f(x) \cdot \frac{g(x+h) - g(x)}{h}}{g(x) \cdot g(x+h)}$$

$$= \frac{g(x) \cdot \boxed{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}} - f(x) \cdot \boxed{\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}}}{g(x) \cdot \lim_{h \to 0} g(x+h)}$$

$$= \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{g(x) \cdot g(x)} = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

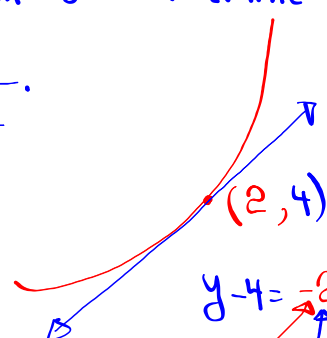
Find the equation of a tan. line at $x=2$

For $f(x) = \frac{2x}{x-1}$.

1) $f(2) = \frac{2(2)}{2-1}$

$= \frac{4}{1}$

$= 4$



$y - 4 = -2(x - 2)$

$$f'(x) = \frac{d}{dx}\left[\frac{2x}{x-1}\right]$$

$$= \frac{2(x-1) - 2x \cdot 1}{(x-1)^2}$$

$$f'(2) = \frac{-2}{(2-1)^2}$$

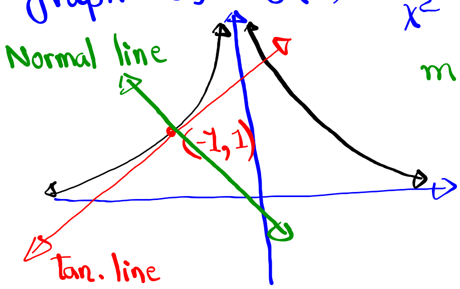
$$= \frac{-2}{1} = -2$$

$$f'(x) = \frac{-2}{(x-1)^2}$$

$y - 4 = -2x + 4$

$\boxed{y = -2x + 8}$

Find equation of the normal line to the graph of $f(x) = \frac{1}{x^2}$ at $x = -1$.



$m_{\text{Normal line}} = \frac{-1}{m_{\text{tan. line}}}$

$f(x) = x^{-2}$

$f'(x) = -2x^{-3}$

$f'(x) = \frac{-2}{x^3}$

$f'(-1) = \frac{-2}{(-1)^3} = 2$

$m_{\text{tan. line}} = 2$

$m_{\text{Normal line}} = \frac{-1}{2}$

$y - y_1 = m(x - x_1)$

$y - 1 = \frac{-1}{2}(x - -1)$

$y = \frac{-1}{2}x - \frac{1}{2} + 1$

$\boxed{y = \frac{-1}{2}x + \frac{1}{2}}$

Class QZ 5

1) Find $\lim_{x\to\infty} \frac{2x - 5}{3x + 2} = \lim_{x\to\infty} \frac{\frac{2x-5}{x}}{\frac{3x+2}{x}} = \lim_{x\to\infty} \frac{2 - \frac{5}{x}}{3 + \frac{2}{x}} = \boxed{\frac{2}{3}}$

2) Find $\lim_{x\to-\infty} \frac{2x - 5}{\sqrt{9x^2+2}} = \lim_{x\to-\infty} \frac{\frac{2x-5}{x}}{\frac{\sqrt{9x^2+2}}{-\sqrt{x^2}}}$

$= \lim_{x\to-\infty} \frac{2 - \frac{5}{x}}{-\sqrt{\frac{9x^2+2}{x^2}}} = -\lim_{x\to-\infty} \frac{2 - \frac{5}{x}}{\sqrt{9 + \frac{2}{x^2}}} = \boxed{\frac{-2}{3}}$